

TOPOLOGY OF GENERALIZED COMPLEX QUOTIENTS

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Dedicated to Prof. Victor Guillemin on the occasion of his seventieth birthday.

ABSTRACT. Consider the Hamiltonian action of a torus on a compact twisted generalized complex manifold M . We first observe that Kirwan injectivity and surjectivity hold for ordinary equivariant cohomology in this setting. Then we prove that these two results hold for the twisted equivariant cohomology as well.

1. INTRODUCTION

Kirwan injectivity and surjectivity are two important results in equivariant symplectic geometry. Recall that for a symplectic manifold (M, ω) , an action by a connected Lie group G on (M, ω) is called Hamiltonian if it is regulated by a moment map $\mu : M \rightarrow \mathfrak{g}^*$ taking values in the dual of the Lie algebra of G . Contracting by $\xi \in \mathfrak{g}$ produces a real valued function $\mu^\xi : M \rightarrow \mathbb{R}$ called a component of the moment map. If G is compact, then for any $\xi \in \mathfrak{g}$, μ^ξ is a Morse-Bott function and may be used to study the equivariant topology of M . We will mostly focus on the case that $G = T$ is a torus.

In [Kir86], using ideas of Atiyah-Bott [AB82], Kirwan demonstrated that a Hamiltonian action on a compact symplectic manifold M is equivariantly formal. In particular, the equivariant cohomology of M with rational coefficients satisfies a noncanonical isomorphism

$$H_G^*(M) \cong H^*(M) \otimes H^*(BG)$$

as graded $H^*(BG)$ -modules, where BG is the classifying space for G . Furthermore, if $G = T$ is a torus, and $i : M^T \hookrightarrow M$ denotes inclusion of the fixed point set, the localization map in equivariant cohomology $i^* : H_T^*(M) \rightarrow H_T^*(M^T)$ is an injection, a result known as Kirwan injectivity. Her proof uses Morse theory of a component μ^ξ of the moment map.

Kirwan also showed that the map $\kappa : H_G(M) \rightarrow H_G(\mu^{-1}(0))$ induced by inclusion is a surjection. This result is known as Kirwan surjectivity and the map κ is known as the Kirwan map. If 0 is a regular value of μ , then $H_G(\mu^{-1}(0)) \cong H(M//G)$, where $M//G = \mu^{-1}(0)/G$ is the symplectic quotient, so $H(M//G)$ is describable as a quotient ring $H_G(M)/\ker(\kappa)$.

Date: August 1, 2008.

Kirwan's original proof of surjectivity involved studying the Morse theory of the norm square of the moment map $\|\mu\|^2$, which has minimum $\mu^{-1}(0)$. In fact $\|\mu\|^2$ is not Morse-Bott, but instead satisfies Kirwan's minimal degeneracy condition, which allows the basic constructions of Morse theory to be carried out.

Modern proofs of Kirwan surjectivity have avoided these technicalities. In [TW98], Tolman and Weitsman computed the kernel of κ for torus actions using the honest Morse-Bott functions μ^ξ , rather than $\|\mu\|^2$, the principle being that the kernel of κ is built up of contributions from each circle in the torus. In [Gol02], Goldin used their ideas to produce a simplified proof of Kirwan surjectivity for torus actions, using circles actions and reduction in stages.

Goldin's proof contains a gap, which was resolved by Ginzburg-Guillemin-Karshon ([GGK02] appendix G). They introduce the notion of a nondegenerate abstract moment map, which abstracts the relevant Morse-theoretic properties from the symplectic case, and prove Kirwan's theorems in this general setting.

In this paper, we generalize Kirwan injectivity and surjectivity to Hamiltonian actions on compact generalized complex manifolds, in the sense of Lin-Tolman [LT05]. Generalized complex (GC) manifolds were introduced by Hitchin in [H02] and developed by Gualtieri in his thesis [Gua03]. They form a common generalization of both complex and symplectic manifolds and so are well suited to the study of Mirror Symmetry and conformal field theory. Generalized complex manifolds can also incorporate a twist by a closed 3-form $H \in \Omega^3(M)$. When H is integral it may be interpreted as the curvature of a gerbe over M and is known in the physics literature as the Neveu-Schwartz 3-form flux.

In the presence of a twisting H , it becomes interesting to study the twisted de Rham cohomology of M , $H(M; H)$, which is defined to be the cohomology of the complex consisting of the usual de Rham forms $\Omega(M)$ with a twisted differential $d + H \wedge$. For example, Gualtieri [Gua04] showed that for an H -twisted generalized Kähler manifold M , $H(M; H)$ inherits a Hodge decomposition and in Kapustin-Li [KL04], $H(M; H)$ is identified as the BRST cohomology of states for the associated conformal field theory.

In [LT05], Lin-Tolman extended the notion of Hamiltonian actions and reduction in symplectic geometry to the realm of generalized complex geometry. In the presence of a twisting 3-form $H \in \Omega^3(M)$, their construction involves a generalized moment map $\mu : M \rightarrow \mathfrak{g}^*$ and a moment 1-form $\alpha \in (\Omega^1(M) \otimes \mathfrak{g}^*)^G$ for which $H + \alpha$ is an equivariantly closed 3-form in the Cartan model (c.f. [GS99]). This construction turns out to be something very natural in physics. It has been shown by Kapustin and Tomasiello [KT06] that the mathematical notion of Hamiltonian actions on twisted generalized Kähler manifolds is in perfect agreement with the physical notion of general $(2, 2)$ gauged sigma models with three-form fluxes.

Inspired by Atiyah-Segal [AS05], the second author in [Lin07] used $H + \alpha$ to define the twisted equivariant cohomology, $H_T(M; \eta + \alpha)$. The basic properties of the twisted equivariant cohomologies were studied in [Lin07] using Hodge theory, especially in the case of Hamiltonian actions on compact generalized Kähler manifolds. In the current paper we study the twisted equivariant cohomology using Morse theory, in the more general case of Hamiltonian actions on compact generalized complex manifolds. The following proposition is crucial to our paper.

Proposition 1.1. *Consider the Hamiltonian action of a (compact) torus T on a compact twisted GC-manifold M with a generalized moment map $\mu : M \rightarrow \mathfrak{t}^*$. Then μ is a nondegenerate abstract moment map in the sense of [GGK02] (see Definition 3.1).*

Proposition 1.1 paraphrases a result from Nitta's very interesting recent work [NY07]. Nitta's result was known by the authors to hold under additional hypotheses but his general result came out as a welcome surprise. Because Nitta's theorem is central to our work, we provide a self-contained proof of Proposition 1.1 in Section 6. Our proof is a variation of Nitta's which has the advantages of being somewhat simpler and of extending to some examples of noncompact manifolds M . One key ingredient in Nitta's proof is the maximum principle for pseudo-holomorphic functions on almost complex manifolds, for which we provide a proof in Appendix A.

In view of the above proposition, results from [GGK02] prove that Kirwan injectivity and surjectivity hold for ordinary equivariant cohomology. In our paper, we use parallel arguments to prove *twisted* versions of equivariant formality, Kirwan injectivity, and Kirwan surjectivity:

Theorem 1.2 (Equivariant formality). *Consider the Hamiltonian action of a compact connected group G on a compact H -twisted generalized complex manifold M . Then we have a non-canonical isomorphism*

$$H_G(M; H + \alpha) \cong H(M; H) \otimes H(BG),$$

where α is the moment one form of the Hamiltonian action.

Theorem 1.3 (Kirwan injectivity). *Let T be a compact torus and let M be a compact H -twisted generalized Hamiltonian T -space with induced equivariant 3-form $H + \alpha$, and let $i : M^T \rightarrow M$ denote the inclusion of the fixed point set. Then the induced map*

$$i^* : H_T(M; H + \alpha) \rightarrow H_T(M^T; H + \alpha) \cong H(M^T; H) \otimes H(BT)$$

is an injection.

Theorem 1.4 (Kirwan surjectivity). *Let M be a compact H -twisted generalized Hamiltonian T -space with induced equivariant 3-form $H + \alpha$ and moment map μ , where T is a compact torus. For $c \in \mathfrak{t}^*$ a regular value of μ we have:*

$$(1.1) \quad H_T(M; H + \alpha) \rightarrow H(\mu^{-1}(c)/T; \tilde{H})$$

is a surjection, where \tilde{H} is the twisting 3-form inherited through reduction.

These results are established more generally for compact nondegenerate abstract moment maps with *compatible* equivariantly closed 3-form. We expect that Kirwan surjectivity remains true for the Hamiltonian action of a compact connected Lie group on a compact twisted generalized complex manifold, and we hope to return to this question in a later work.

Non-symplectic examples of Hamiltonian torus actions on generalized complex manifolds to which our results may be applied have been constructed in [Lin07] and [Lin07b]. New examples constructed using surgery on toric varieties will be included in a forthcoming paper by the authors. We would also like to mention that a stronger version of Theorem 1.2 was previously proven for the case of Hamiltonian action of a compact Lie group on a generalized Kähler manifold in [Lin07], using the $\partial\bar{\partial}$ lemma and generalized Hodge theory.

We discuss now one possible application of our results. Suppose (M, \mathcal{J}) is a twisted generalized complex manifold with a Hamiltonian G action, and suppose L is the $\sqrt{-1}$ -eigenbundle of \mathcal{J} . Then L has a natural Lie algebroid structure, c.f. [Gua03]. Moreover, the existence of a Hamiltonian G action induces a Lie algebra map $\mathfrak{g} \rightarrow C^\infty(L)$. So there is an equivariant version of the Lie algebroid cohomology associated to the Hamiltonian G action, in the sense of [BCRR05]. The twisted equivariant cohomology studied in the current paper is closely related to the equivariant Lie algebroid cohomology. Indeed, they are canonically isomorphic to each other if M is a generalized Calabi-Yau manifold satisfying the $\bar{\partial}\partial$ -lemma. It is well known that information on the deformation of generalized complex structures is contained in the Lie algebroid cohomology of L . Therefore, the results established in this paper may indicate a close relationship between the deformation theory of the generalized complex manifold M and that of its generalized complex quotients.

The layout of the paper is as follows. Section 2 reviews twisted equivariant cohomology and proves a few lemmas for later use. Section 3 uses Morse theory to prove twisted Kirwan injectivity and surjectivity for nondegenerate abstract moment maps. Section 4 gives a quick review of generalized complex geometry. Section 5 recalls the definition of generalized moment maps and proves Proposition 1.1. Section 6 establishes the main results of this paper, namely, Theorem 1.2, 1.3, 1.4. Appendix A proves the maximum principle for pseudoholomorphic functions on almost complex manifolds. Appendix B establishes a key Lemma about nondegenerate abstract moment maps postponed from §3. Appendix C compares several

versions of twisted equivariant cohomology existed in the literature. Appendix D collects some commutative algebra results that we make frequent use of throughout, but particularly in Section 2.

Acknowledgement: The authors would like to thank Marco Gualtieri, Yael Karshon, Eckhard Meinrenken, Paul Selick, and Ping Xu for useful discussions. Y.L. is grateful to Lisa Jeffrey for providing him a Postdoctoral Fellowship at the University of Toronto, where he started his work on the equivariant cohomology theory of GC manifolds.

2. REVIEW OF TWISTED EQUIVARIANT COHOMOLOGY

In this section we review twisted equivariant cohomology, as developed in Atiyah-Segal [AS05], Hu-Urbe [HuU06], Freed-Hopkins-Teleman [FHT02] and Lin [Lin07].

2.1. Definitions. Let G be a compact connected Lie group with Lie algebra \mathfrak{g} and dual \mathfrak{g}^* . For M a smooth G -manifold, we denote by ξ_M the vector field on M generated by $\xi \in \mathfrak{g}$. The equivariant de Rham complex $(\Omega_G(M), d_G)$ is a differential graded (super)commutative algebra associated to the G -manifold M . Here,

$$\Omega_G(M) = (\Omega(M) \otimes S\mathfrak{g}^*)^G$$

is the space of polynomial functions on \mathfrak{g} taking values in the space of differential forms $\Omega(M)$, which are equivariant under the induced G -action on $\Omega(M)$ and the adjoint action on the symmetric algebra $S\mathfrak{g}^*$, and d_G is defined by extending linearly the formula

$$(d_G(\sigma \otimes P))(\xi) = d\sigma \otimes P(\xi) - \iota_{\xi_M} \sigma \otimes P(\xi)$$

where $\sigma \in \Omega(M)$, $P \in S\mathfrak{g}^*$, and $\xi \in \mathfrak{g}$. It comes equipped with a grading

$$\Omega_G^n(M) = \bigoplus_k (\Omega^{n-2k}(M) \otimes S^k \mathfrak{g}^*)^G.$$

The equivariant de Rham complex computes the (Borel) equivariant cohomology of M with real coefficients (we refer to [GS99] for more details).

Example 2.1. When G is trivial $(\Omega_G(M), d_G)$ is the usual de Rham complex.

Example 2.2. In the special case that $G = T$ is a compact torus with lie algebra \mathfrak{t} , T acts trivially on \mathfrak{t} we have:

$$\Omega_T(M) = \Omega(M)^T \otimes S\mathfrak{t}^*$$

where $\Omega(M)^T$ is the space of T -invariant differential forms.

Let $\hat{\Omega}_G(M)$ denote the direct product $\prod \Omega_G^i(M)$. The differential d_G extends in a natural way to $\hat{\Omega}_G(M)$ and we adopt the convention that $H_G(M)$ is defined to be the cohomology of the complex $(\hat{\Omega}_G(M), d_G)$ (where we have abusively reused d_G to denote its extension to $\hat{\Omega}_G(M)$). It follows that

$$(2.1) \quad H_G(M) := \prod_{i=0}^{\infty} H_G^i(M)$$

as opposed to the more conventional direct sum. In the untwisted setting this is not a serious modification, but once twisting is introduced the direct product is much easier to work with. In this context, the equivariant cohomology of a point is $(\hat{\mathcal{S}}\mathfrak{g}^*)^G$, the ring of G -invariant formal power series on \mathfrak{g} .

Given a d_G -closed 3-form, $\eta \in \Omega_G^3(M)$, we define a twisted differential

$$d_{G,\eta} = d_G + \eta \wedge$$

on $\hat{\Omega}_G(M)$. Because η is closed and of odd degree, it follows that $d_{G,\eta}^2 = 0$ and we define the **η -twisted equivariant cohomology**

$$H_G(M; \eta) = \ker d_{G,\eta} / \operatorname{im} d_{G,\eta}.$$

Because $d_{G,\eta}$ is an odd operator, $H(M; \eta)$ inherits a \mathbb{Z}_2 -grading from the \mathbb{Z} -grading on $\Omega_G(M)$. Because $d_{G,\eta}$ is usually not a derivation, $H_G(M; \eta)$ is usually not a ring, but is instead a module for the untwisted equivariant cohomology ring $H_G(M)$ and hence also for $\hat{\mathcal{S}}\mathfrak{g}^*$.

Remark 2.3. The cochain complex we are using to define twisted cohomology differs from those found in [FHT02] and [HuU06], but gives rise to a naturally isomorphic cohomology theory (see Appendix C).

Remark 2.4. We show in Appendix C that for a compact manifold M the completion of twisted equivariant cohomology is obtained by extension of scalars from the uncompleted version, i.e.

$$H_G(M; \eta) \cong H(\Omega_G(M), d_{G,\eta}) \otimes_{(\mathcal{S}\mathfrak{g}^*)^G} (\hat{\mathcal{S}}\mathfrak{g}^*)^G$$

Example 2.5. Suppose G acts trivially on M . Then any d -closed 3-form $\eta \in \Omega^3(M)$ determines a d_G -closed 3-form $\eta \otimes 1 \in \Omega_G^3(M)$. In this case it is easy to see that $H_G(M; \eta \otimes 1) \cong H(M; \eta) \otimes (\hat{\mathcal{S}}\mathfrak{g}^*)^G$ canonically as $(\hat{\mathcal{S}}\mathfrak{g}^*)^G$ -modules.

Example 2.6. More generally, let G act on M such that a normal subgroup $H \subset G$ acts trivially. Then there is an induced G/H action on M and a chain isomorphism $\Omega_{G/H}(M) \otimes (\mathcal{S}\mathfrak{h}^*)^H \cong \Omega_G(M)$. If $\eta \in \Omega_{G/H}^3(M)$ is $d_{G/H}$ -closed, then

$$H_G(M; \eta \otimes 1) \cong H_{G/H}(M; \eta) \otimes (\hat{\mathcal{S}}\mathfrak{h}^*)^H$$

canonically.

We may consider a more general class of twisted complexes using the notion of differential graded modules. Let $(C^*, \delta) = (\oplus_{k \geq 0} C^k, \delta)$ be a cochain complex graded by the nonnegative integers. We say that (C^*, δ) is a (left) $(\Omega_G^*(M), d_G)$ -module, or simply a $\Omega_G(M)$ -module, if C^* is a graded module of the graded algebra $\Omega_G^*(M)$ and for all $\alpha \in \Omega_G(M)$ of pure degree and $x \in C^*$, the differential satisfies the identity:

$$\delta(\alpha \wedge x) = d_G(\alpha) \wedge x + (-1)^{\deg \alpha} \alpha \wedge \delta(x).$$

The differential δ extends naturally to a differential on $\hat{C} := \prod_i C^i$, which by abuse of notation we also call δ . A closed 3-form $\eta \in \Omega_G(M)$ determines a twisted differential $\delta_\eta := \delta + \eta \wedge$ on \hat{C} and we define

$$H_G(C^*; \eta) = \ker \delta_\eta / \text{im} \delta_\eta.$$

The module structure descends to make $H_G(C^*; \eta)$ a \mathbb{Z}_2 -graded module for $H_G(M)$.

Example 2.7. Let $i : A \subset N$ a pair of embedded submanifolds of M preserved by G . We use the algebraic mapping cone to define the differential graded complex $(\Omega_G^*(N, A), \delta)$ by

$$\Omega_G^n(N, A) = \Omega_G^{n+1}(N) \oplus \Omega_G^n(A)$$

with differential $\delta(n, a) = (-d_G(n), d_G(a) + i^*(n))$. Then $\Omega_G(N, A)$ is a $(\Omega_G(M), d_G)$ -module under the action $x \wedge (n, a) = (x \wedge n, x \wedge a)$. For $\eta \in \Omega_G^3(M)$ closed, we will use notation:

$$H_G(N, A; \eta) = H(\Omega_G(N, A); \eta)$$

Notice that $H_G(N, A; \eta) = H_G(N, A; j^* \eta)$, where $j : N \rightarrow M$ is the inclusion map.

2.2. Basic Properties. Twisted cohomology is invariant under quasiisomorphism.

Proposition 2.8. *Let $\phi : (C^*, \delta) \rightarrow (D^{*+n}, \delta')$ be a degree n quasiisomorphism of $(\Omega_G(M), d_G)$ -modules. Then the induced map $H_G^*(C^*; \eta) \rightarrow H_G^{*+n}(D^*; \eta)$ is a degree $n \bmod 2$ isomorphism for all d_G -closed $\eta \in \Omega_G^3(M)$.*

Proof. Using the algebraic mapping cone construction, it suffices to prove that if (C^*, δ) is acyclic, then so is (\hat{C}^*, δ_η) .

Let $c = c_i + c_{i+1} + c_{i+2} + \dots \in \hat{C}^*$ be δ_η -closed, where $c_k \in C^k$. Then necessarily $\delta(c_i) = 0$. By acyclicity, there exists $b_{i-1} \in C^{i-1}$ such that $\delta(b_{i-1}) = c_i$, so

$$c - \delta_\eta(b_{i-1}) = c_{i+1} + (c_{i+2} - \eta \wedge b_{i-1}) + \dots$$

has lowest degree term lying in C^{i+1} . Iterating the process, we can construct $b = b_{i-1} + b_i + \dots$ satisfying $\delta_\eta(b) = c$. \square

It follows that many important properties of untwisted equivariant cohomology, such as homotopy invariance and excision, extend to twisted cohomology.

Proposition 2.9. *Let G be compact connected with maximal torus T and Weyl group $W = N(T)/T$. For any G -manifold M and twisting $\eta \in \Omega_G^3(M)$, we have a natural isomorphism*

$$H_G(M; \eta) \cong H_T(M; \eta')^W$$

where $\eta' \in \Omega_T(M)^W$ is the image of η under the map $\Omega_G(M) \rightarrow \Omega_T(M)$ induced by restricting the action.

Proof. The map $\Omega_G(M) \rightarrow \Omega_T(M)$ restricts to a quasiisomorphism $\Omega_G(M) \rightarrow \Omega_T(M)^W$ which is also a $\Omega_G(M)$ -module homomorphism in the obvious way. Thus by Proposition 2.8,

$$H_G(M; \eta) \cong H_T(\Omega_T(M)^W; \eta) \cong H_T(M; \eta')^W$$

□

This result helps justify our later focus on torus actions.

Recall that in untwisted equivariant cohomology, we have the isomorphism

$$\phi : H_G(M) \cong H(M/G),$$

provided that the action of G on M is free. We have the following generalization.

Proposition 2.10. ([Lin07], A.4.) *Let M be a smooth G -manifold upon which G acts freely and suppose $\dim H(M) < \infty$. For d_G -closed $\eta \in \Omega_G^3(M)$ we have isomorphisms*

$$H_G(M; \eta) \cong H(M/G; \bar{\eta})$$

where $\bar{\eta} \in \Omega^3(M/G)$ satisfies $\phi([\eta]) = [\bar{\eta}] \in H(M/G)$.

Given a short exact sequence $0 \rightarrow C^* \rightarrow D^* \rightarrow E^* \rightarrow 0$ of $\Omega_G(M)$ -modules, twisting by η gives rise to a six term exact sequence in the twisted cohomology.

Example 2.11. Recall the notation of Example 2.7. The pair $i : A \hookrightarrow N$ gives rise to a short exact sequence of $\Omega_G(M)$ -modules, $0 \rightarrow \Omega_G(A) \rightarrow \text{Cyl}(i^*) \rightarrow \Omega_G(N, A) \rightarrow 0$, where the algebraic mapping cylinder, $\text{Cyl}(i^*)$, is naturally quasiisomorphic to $\Omega_G(N)$. We obtain a six term exact sequence:

$$\begin{array}{ccccc} H_G^0(N, A; \eta) & \longrightarrow & H_G^0(N; \eta) & \longrightarrow & H_G^0(A; \eta) \\ \uparrow & & & & \downarrow \\ H_G^1(A; \eta) & \longleftarrow & H_G^1(N; \eta) & \longleftarrow & H_G^1(N, A; \eta) \end{array}$$

The next lemma shows that up to (noncanonical) isomorphism, the η -twisted equivariant cohomology depends only on the cohomology class $[\eta] \in H_G^3(M)$.

Lemma 2.12. *Let $b \in \Omega_G^2(M)$ be an equivariant 2-form and let $\exp(b) = \sum_{i=0}^{\infty} b^i/i!$. Then for any $(\Omega_G(M), d_G)$ -module (C^*, δ) , wedging by $\exp(b)$ determines an isomorphism of chain complexes,*

$$\exp(b) \wedge (\cdot) : (\hat{C}, \delta_{(\eta+d_G b)}) \rightarrow (\hat{C}, \delta_\eta)$$

which in particular determines an isomorphism $H_G(C^; \eta + d_G b) \cong H_G(C^*; \eta)$.*

Proof. The map $\exp(b) \wedge (\cdot) : \hat{C} \rightarrow \hat{C}$ is certainly even and linear. The equations $\exp(b) \wedge \exp(-b) = \exp(-b) \wedge \exp(b) = \text{id}_{\hat{C}}$ imply that $\exp(b) \wedge (\cdot)$ is an isomorphism of vector spaces. Finally, for $\alpha \in \hat{C}$ we have

$$\begin{aligned} \delta_\eta(\exp(b) \wedge \alpha) &= d_G(\exp(b)) \wedge \alpha + \exp(b) \wedge \delta_\eta \alpha \\ &= \exp(b) \wedge d_G b \wedge \alpha + \exp(b) \wedge \delta_\eta \alpha = \exp(b) \delta_{(\eta+d_G b)} \alpha \end{aligned}$$

so $\exp(b) \wedge (\cdot)$ respects differentials. \square

We may now state and prove the Thom isomorphism, which is due to Hu-Urbe [HuU06].

Proposition 2.13. *Let $\pi : E \rightarrow N$ be an orientable real vector bundle of rank r , let $i : N \rightarrow E$ denote inclusion as the zero section. Let G be a compact torus acting on E by bundle automorphisms, inducing an action on N . Let $\eta \in \Omega_G^3(E)$ be a d_G -closed form and let $\tau \in \Omega_G^r(E, E - N)$ be a d_G -closed form representing the usual equivariant Thom class (c.f. [GS99]). Then the composition:*

$$H_G(N; \eta) \xrightarrow{\pi^*} H_G(E; \pi^* i^* \eta) \xrightarrow{\wedge \tau} H_G(E, E - N; \pi^* i^* \eta) \xrightarrow{\exp(b) \wedge} H_G(E, E - N; \eta)$$

is an isomorphism of degree $(r \bmod 2)$, where $b \in \Omega_G^2(E)$ satisfies $d_G(b) = \pi^ i^*(\eta) - \eta$.*

Proof. By homotopy invariance π^* is an isomorphism. The map $\wedge \tau : (\Omega_G(E), d_G) \rightarrow (\Omega_G(E, E - N), d_G)$ is a $\Omega_G(E)$ -module morphism and induces a degree r isomorphism $H_G^*(E) \cong H_G^{*+r}(E, E - N)$, so by Proposition 2.8

$$H_G(E; \eta) \xrightarrow{\wedge \tau} H_G(E, E - N; \pi^* i^* \eta)$$

is also an isomorphism. Finally, $\exp(b) \wedge$ is an isomorphism by Lemma 2.12. \square

The equivariant Euler class plays the same role for twisted equivariant cohomology as it does for untwisted equivariant cohomology.

Lemma 2.14. *Let $\pi : E \rightarrow N$ satisfy the hypotheses and notation of Proposition 2.13, and let $\text{Eul}_G(E) \in H_G^r(N)$ denote the equivariant Euler class of E . Then the following diagram is commutative:*

$$(2.2) \quad \begin{array}{ccc} H_G(E, E - N; \eta) & \xrightarrow{j} & H_G(E; \eta) \\ \uparrow \phi & & \downarrow i^* \\ H_G(N; \eta) & \xrightarrow{\cup \text{Eu}_G(E)} & H_G(N; \eta) \end{array}$$

where ϕ is the Thom isomorphism of Proposition 2.13, and j is induced by the inclusion map of forms.

Proof. The map $i^* \circ j \circ \phi$ is induced by a map of forms $h : \Omega_G(N) \rightarrow \Omega_G(N)$ defined by $h(\alpha) = i^*(\exp(b) \wedge \pi^*(\alpha) \wedge \tau)$ where τ is a form representing the equivariant Thom class and $b \in \Omega_G^2(N)$ satisfies $d_G(b) = \eta - \pi^*(i^*(\eta))$. We may choose b so that $i^*(b) = 0$, because if it doesn't we can replace it by $b - \pi^*(i^*(b))$. Thus $h(\alpha) = i^*(\exp(b) \wedge \pi^*(\alpha) \wedge \tau) = i^*(\exp(b)) \wedge \alpha \wedge i^*(\tau) = \alpha \wedge i^*(\tau)$. Because $i^*(\tau)$ represents the equivariant Euler class, this completes the proof. \square

Hu and Uribe go on to prove the following twisted version of the localization theorem.

Theorem 2.15. ([HuU06]) *Let T be a compact torus acting on a smooth, closed manifold M and let $i : M^T \hookrightarrow M$ denote the inclusion of the fixed point set. Then for any d_T -closed 3-form $\eta \in \Omega_T^3(M)$, the kernel and cokernel of the induced map $i^* : H_T(M; \eta) \hookrightarrow H_T(M^T; \eta)$ are $\hat{S}t^*$ -torsion.*

2.3. Spectral Sequences. We now consider two spectral sequences associated filtrations of the complex $(\hat{\Omega}_G(M), d_\eta)$, both of which converge strongly to $H(M; \eta)$ (see Appendix C for an explanation of convergence properties).

First consider the filtration of $(\hat{\Omega}_G(M), d_{G, \eta})$

$$(2.3) \quad F^p = F^p \hat{\Omega}_G(M) := \prod_{k \geq p} \Omega_G^k(M)$$

which satisfies $d_{G, \eta}(F^p) \subset F^{p+1}$. The resulting spectral sequence (E_r^*, d_r) satisfies $E_1^p \cong E_2^p \cong H_G^p(M)$ the untwisted cohomology, while d_2 is the wedging map $\eta \wedge (\cdot) : H_G^*(M) \rightarrow H_G^*(M)$. Thus by Proposition 2.12, this spectral sequence collapses at E_1 if and only if η is cohomologous to zero. In particular, if $\dim H(M) < \infty$ then in the nonequivariant case:

$$(2.4) \quad \dim H(M; \eta) \leq \dim H(M)$$

with equality if and only if η is d -exact.

Now consider a different filtration:

$$(2.5) \quad L^p = L^p \hat{\Omega}_G(M) := \prod_{k \geq p} (\Omega(M) \otimes S^k \mathfrak{g}^*)^G$$

This gives rise to a spectral sequence (E_r^*, d_r) of $(\hat{S}\mathfrak{g}^*)^G$ -modules satisfying $E_1^p \cong H(M; \eta(0)) \otimes (S^p \mathfrak{g}^*)^G$, where $\eta(0)$ be the ordinary 3-form obtained by evaluating η at $0 \in \mathfrak{g}$.

Definition 2.16. We say that a G -manifold M is η -equivariantly formal if the spectral sequence defined above collapses at E_1 . In this case $H_G(M; \eta)$ is non-canonically isomorphic to $H(M; \eta(0)) \otimes (\hat{S}\mathfrak{g}^*)^G$ as a module over $(\hat{S}\mathfrak{g}^*)^G$.

Notice that the quotient complex $\hat{\Omega}_G(M)/L^1\hat{\Omega}_G(M)$ is canonically isomorphic to $\Omega(M)$. This gives rise to a natural map $H_G(M; \eta) \rightarrow H(M; \eta(0))$ for all twistings η . We have a version of the Leray-Hirsch theorem in this context.

Proposition 2.17. The G -manifold M is η -equivariantly formal if and only if the natural map $H_G(M; \eta) \rightarrow H(M; \eta(0))$ is surjective.

Proof. The spectral sequence associated to the filtration collapses at page E_1 if and only if the injections $L^{p+1} \hookrightarrow L^p$ induce injections in cohomology $H(L^{p+1}\Omega_G(M); d_{G,\eta}) \hookrightarrow H(L^p\Omega_G(M); d_{G,\eta})$ for all p . By the associated six term exact sequence, is true if and only if $H(L^p; d_{G,\eta}) \rightarrow H(L^p/L^{p+1}; d_{G,\eta})$ is surjective. Of course $H(L^p/L^{p+1}; d_{G,\eta}) = E_1^p \cong H(M; \eta(0)) \otimes (S^p \mathfrak{g}^*)^G$.

Collecting together, we see that M is equivariantly formal if and only if the natural maps

$$\pi_p : H(L^p; d_{G,\eta}) \rightarrow H(M; \eta(0)) \otimes (S^p \mathfrak{g}^*)^G$$

are surjective for all p . When $p = 0$, π_0 is exactly $H_G(M; \eta) \rightarrow H(M; \eta(0))$ proving one direction of the equivalence. The opposite direction follows by noting that for $\sigma \in (S^p \mathfrak{g}^*)^G$ and $d_{G,\eta}$ -closed $\phi \in \Omega_G(M)$, we have $\pi_p(\phi\sigma) = \pi_0(\phi) \otimes \sigma$. \square

Proposition 2.18. A G -manifold M is η -equivariantly formal if and only if M it is η' -equivariantly formal as a T -manifold under the restricted maximal torus action, where η' is the image of η under the induced map $\Omega_G^3(M) \rightarrow \Omega_T^3(M)$.

Proof. We use the criterion of Proposition 2.17 The natural map $\phi : H_G(M; \eta) \rightarrow H(M; \eta(0))$ factors through the natural map ϕ' via

$$H_G(M; \eta) \rightarrow H_T(M; \eta') \xrightarrow{\phi'} H(M; \eta'(0)).$$

Thus if ϕ is surjective, so must ϕ' . On the other hand, the map ϕ' is W equivariant, where the action of the Weyl group W on $H(M; \eta(0))$ is induced by $N(T)$ action restricted from G . Since this action is isotropically trivial, we find that if ϕ' is invariant under the W action, so is the restricted map

$$H_G(M; \eta) \cong H_T(M; \eta')^W \rightarrow H(M; \eta(0))$$

completing the proof. \square

It is worth noting that a G -manifold that is equivariantly formal for $\eta = 0$ may fail to be formal for $\eta \neq 0$.

Example 2.19. Let $U(1)$ act trivially on S^1 . Then $H_{U(1)}(S^1) \cong H(S^1) \otimes \mathbb{R}[x]$, where $0 \neq x \in \mathfrak{u}(1)^*$. Choose a twisting η satisfying $0 \neq [\eta] \in H^1(S^1) \otimes x$. Then $H(S^1; \eta(0)) \cong H(S^1)$ because $[\eta(0)] = 0$, while $H_{U(1)}(M; \eta) = 0$. The second assertion here follows from the $\{L^p\}$ spectral sequence where $E_1 \cong H(S^1) \otimes \mathbb{R}[x]$ with differential d_1 defined by wedging by $[\eta]$ so that $H(E_1, d_1) = E_2 = 0$.

3. MORSE THEORY

Throughout this section, unless otherwise stated, T is a compact torus with Lie algebra \mathfrak{t} acting on a closed smooth manifold M . Recall the following definition from [GGK02]

Definition 3.1. *An nondegenerate abstract moment map $\mu : M \rightarrow \mathfrak{t}^*$ is a smooth, equivariant map $\mu : M \rightarrow \mathfrak{t}^*$ such that for every vector $\xi \in \mathfrak{t}$,*

- (1) $\text{Crit}(\mu^\xi) = \{\xi_M = 0\}$, and
- (2) $\mu^\xi : M \rightarrow \mathbb{R}$ is a Morse-Bott function.

Definition 3.1 is an abstraction of the Morse theoretic properties of symplectic moment maps which are responsible for results such as Kirwan injectivity and surjectivity, as well as convexity (c.f. [GGK02]). In [NY07] Nitta actually proved that the components of moment map for Hamiltonian torus actions on compact generalized complex manifolds are abstract nondegenerate moment maps (see also §5). Thus it follows that Kirwan injectivity and surjectivity for the usual equivariant cohomology must hold for GC-Hamiltonian actions.

To prove twisted versions of these theorems, we must impose a compatibility condition on the twisting 3-form.

Definition 3.2. *Let $\eta \in \Omega_T^3(M) = \Omega^3(M)^T \oplus (\Omega^1(M)^T \otimes \mathfrak{t}^*)$ and let η^1 denote the component of η lying in $\Omega^1(M)^T \otimes \mathfrak{t}^*$. We say that η is **compatible** if for all $p \in M$, $\ker(\eta_p^1) \supseteq \mathfrak{t}_p$, where η^1 is regarded as a linear map $\eta^1 : \mathfrak{t} \rightarrow \Omega^1(M)^T$ and \mathfrak{t}_p is the Lie algebra of the isotropy subgroup of the point $p \in M$.*

Our goal in this section is to prove the following two results:

Theorem 3.3. *[Kirwan Injectivity] Let M be a smooth, compact T -manifold with abstract moment map $\mu : M \rightarrow \mathfrak{t}^*$, and compatible equivariantly closed 3-form $\eta \in \Omega_T^3(M)$. Then M is **η -equivariantly formal**. In particular, the localization map $i^* : H_T(M; \eta) \rightarrow H_T(M^T; \eta)$ is injective and*

$$H_T(M; \eta) \cong H(M; \eta(0)) \otimes \hat{S}(\mathfrak{t}^*)$$

noncanonically as $\hat{S}(\mathfrak{t}^)$ -modules.*

Theorem 3.3 may be generalized to noncompact M using the weaker hypothesis that the fixed point set M^T is compact and that some nonzero component of the moment map $\mu^\xi : M \rightarrow \mathbb{R}$ is proper and bounded below. Working in such generality is cumbersome, so we stick with compact M .

Theorem 3.4. [Kirwan Surjectivity] *Let M be a smooth, compact T -manifold with abstract moment map $\mu : M \rightarrow \mathfrak{t}^*$, and compatible equivariantly closed 3-form $\eta \in \Omega_T(M)$. Suppose M admits an invariant almost complex structure. Then the map in equivariant cohomology induced by inclusion of the zero level set:*

$$H_T(M; \eta) \rightarrow H_T(\mu^{-1}(0); \eta)$$

is a surjection. In the event that T acts freely on $\mu^{-1}(0)$ then $H_T(\mu^{-1}(0); \eta) \cong H(\mu^{-1}(0)/T; \bar{\eta})$ as explained in Proposition 2.10.

As before, Theorem 3.4 may be generalized to include some examples of noncompact manifolds but for the sake of simplicity we work with compact M . In our proof of Theorem 3.4 we found it necessary to require a invariant almost complex structure, though we suspect the theorem holds without this additional hypothesis. The presence of an invariant almost complex structure in the case of a GC Hamiltonian actions was proven by Nitta [NY07] and played an important part in his work.

Let $f : M \rightarrow \mathbb{R}$ be a smooth function. We denote the critical set of f by $\text{Crit}(f) = \{x \in M \mid df_x = 0\}$. For $x \in \text{Crit}(f)$, the Hessian $\text{Hess}_x(f) : T_x M \rightarrow T_x^* M$ is the symmetric linear map defined by the formula

$$\langle \text{Hess}_x(f)(v), w \rangle = w \cdot L_{\tilde{v}} f,$$

where $v, w \in T_x M$, \tilde{v} is any vector field satisfying $\tilde{v}_x = v$, L is the Lie derivative and \langle, \rangle is the pairing between $T_x M$ and $T_x^* M$. The Hessian is more often defined as the quadratic form $\langle \text{Hess}_x(\cdot), \cdot \rangle$, but the definition as a linear map is more convenient for us.

Definition 3.5. *Let M be a smooth, closed manifold. A smooth function $f : M \rightarrow \mathbb{R}$ is called **Morse-Bott** if the connected components of $\text{Crit}(f) = \{x \in M \mid df_x = 0\}$ are closed submanifolds of M and for all $x \in \text{Crit}(f)$ the kernel of the Hessian satisfies $\ker(\text{Hess}_x(f)) = T_x \text{Crit}(f)$.*

Let $\{C_i \mid i \in 0, 1, 2, \dots, n\}$ be the set of connected components of $\text{Crit}(f)$. The function f is constant on each component C_i and we define $c_i := f(C_i) \in \mathbb{R}$. We will assume for simplicity of exposition that $c_i = c_j$ if and only if $i = j$, though all the proofs can be adapted to work without this assumption. We choose the indexing $i = 0, 1, \dots, n$ so that $c_i < c_j$ if and only if $i < j$.

Choose a Riemannian metric g on M . Using g to identify $TM \cong T^*M$, we may regard $\text{Hess}_x(f)$ as an automorphism of $T_x M$ for $x \in \text{Crit}(f)$. Because it is symmetric, $\text{Hess}_x(f)$ is diagonalizable with real eigenvalues. We define the negative normal bundle ν_i of C_i by setting $\nu_{i,x}$ to equal the sum of negative eigenspaces of $\text{Hess}_x(f)$. Up to isomorphism, ν_i is independent of the choice of g . We call the rank of ν_i the **index** of C_i and denote it $\lambda(i)$. In the presence of a compact torus T -action on M leaving f and g invariant, the ν_i become equivariant vector bundles over C_i .

Let $M_t := f^{-1}((-\infty, t])$. If the interval $[s, t]$ contains no critical values for f , the inclusion $M_s \hookrightarrow M_t$ is a homotopy equivalence. In particular, if a

torus T acts on M leaving f invariant and $\eta \in \Omega_T^3(M)$ is a closed equivariant 3-form, then $H_T(M_t; \eta) \cong H_T(M_s; \eta)$.

Thus for some $\epsilon > 0$ sufficiently small, we obtain for each critical value c_i a six term exact sequences:

$$(3.1) \quad \begin{array}{ccccc} H_T^0(M_{c_i+\epsilon}, M_{c_i-\epsilon}; \eta) & \longrightarrow & H_T^0(M_{c_i+\epsilon}; \eta) & \longrightarrow & H_T^0(M_{c_i-\epsilon}; \eta) \\ \uparrow & & & & \downarrow \\ H_T^1(M_{c_i-\epsilon}; \eta) & \longleftarrow & H_T^1(M_{c_i+\epsilon}; \eta) & \longleftarrow & H_T^1(M_{c_i+\epsilon}, M_{c_i-\epsilon}; \eta) \end{array}$$

and canonical isomorphisms $H_T(M_{c_i+\epsilon}) \cong H_T(M_{c_{i+1}-\epsilon})$.

Using excision and the Thom isomorphism, we obtain isomorphisms:

$$(3.2) \quad H_T^*(M_{c_i+\epsilon}, M_{c_i-\epsilon}; \eta) \cong H_T^*(\nu_i, \nu_i - 0; \eta) \cong H_T^{*+\lambda(i)}(C_i; \eta)$$

where the superscript grading is taken mod 2.

Definition 3.6. A T -invariant Morse-Bott function f is called η -equivariantly perfect if the vertical arrows in (3.1) are zero for all critical values c_i .

An important consequence is that $\oplus_i H_T(M_{c_i+\epsilon}, M_{c_i-\epsilon}; \eta)$ is isomorphic to an associated graded object of $H_T(M; \eta)$.

Proposition 3.7. Suppose that f is bounded below, η -equivariantly perfect Morse-Bott function on M and that the negative normal bundles are all orientable. Then there is an isomorphism of $\hat{S}t^*$ -modules

$$\text{gr}(H_T^*(M; \eta)) \cong \oplus_i H_T^{*+\lambda(i)}(C_i; \eta)$$

where $\text{gr}(H_T(M; \eta))$ is the associated graded ring determined by the topological filtration M_s of M and $\lambda(i) \in \{0, 1\}$ is the index of C_i mod 2.

Proof. Because f is η -equivariantly perfect, the exact sequence (3.1) decomposes into exact sequences

$$(3.3) \quad 0 \rightarrow H_T(M_{c_i+\epsilon}, M_{c_i-\epsilon}) \rightarrow H_T(M_{c_i+\epsilon}; \eta) \rightarrow H_T(M_{c_i-\epsilon}; \eta) \rightarrow 0$$

It follows that $\text{gr}(H_T(M; \eta)) \cong \oplus_i H_T(M_{c_i+\epsilon}, M_{c_i-\epsilon})$. Applying (3.2) completes the proof. \square

It was noticed by Atiyah and Bott that an invariant Morse-Bott function can sometimes be shown to be equivariantly perfect using only negative normal bundle data as follows. If the negative normal bundle is orientable, we construct a commutative diagram:

$$\begin{array}{ccccc}
H_T(M_{c+\epsilon}, M_{c-\epsilon}; \eta) & \xrightarrow{j} & H_T(M_{c+\epsilon}; \eta) & \longrightarrow & H_T(M_{c-\epsilon}; \eta) \\
\downarrow \cong & & \downarrow & & \\
(3.4) \quad H_T(\nu_i, \nu_i - 0; \eta) & \longrightarrow & H_T(\nu_i; \eta) & & \\
\uparrow \cong & & \downarrow \cong & & \\
H_T(C_i; \eta) & \xrightarrow{\cup \text{Eul}_T(\nu_i)} & H_T(C_i; \eta) & &
\end{array}$$

where the upper square is excision and the bottom square is from Lemma 2.14. If $\cup \text{Eul}_T(\nu_i) : H_T(C_i; \eta) \rightarrow H_T(C_i; \eta)$ is injective then j must also be injective. We obtain the self perfecting principle:

Lemma 3.8. *Suppose that for all critical sets C_i , $\text{Eul}_T(\nu_i)$ is not a zero divisor for $H_T(C_i; \eta)$, i.e. for all $\alpha \in H_T(C_i; \eta)$, we have $\alpha \cup \text{Eul}_T(\nu_i) = 0$ if and only if $\alpha = 0$. Then f is η -equivariantly perfect.*

Atiyah and Bott discovered a simple criterion implying that $\text{Eul}_T(\nu_i)$ is not a zero divisor in the nontwisted setting. We adapt their proof to the twisted case.

Lemma 3.9. *Let $\nu \rightarrow N$ be a T -equivariant oriented vector bundle over a compact manifold N and suppose there exists a subtorus $S \subset T$ such that ν^S is exactly the zero section of ν . Let $\eta \in \Omega_{T/S}(N) \hookrightarrow \Omega_T(N)$ under the natural inclusion (see Example 2.6). Then $\text{Eul}_T(\nu)$ is not a zero divisor for $H_T(N; \eta)$.*

Proof. By Example 2.6 the untwisted equivariant cohomology satisfies

$$H_T(N) \cong H_{T/S}(N) \otimes \hat{S}(\mathfrak{s}^*).$$

It was shown in [AB82] §13, that the equivariant Euler class $\text{Eul}_T(\nu) \in H_T(N)$ satisfies

$$\text{Eul}_T(\nu) = 1 \otimes \beta_0 + \text{positive degree terms in } H_{T/S}^*(N)$$

where $\beta_0 \in S(\mathfrak{s}^*)$ is nonzero.

Also by Example 2.6, we have a natural isomorphism

$$H_T(N; \eta) \cong H_{T/S}(N; \eta) \otimes \hat{S}(\mathfrak{s}^*)$$

The ideal $I = \prod_{k>0} H_{T/S}^k(N)$ is the Jacobson ideal of $H_{T/S}(N)$, so by Nakayama's Lemma the filtration $\{F^p\}$ of $H_T(N; \eta)$ defined by $F^p := I^p \cup H_{T/S}(N; \eta) \otimes \hat{S}(\mathfrak{s}^*)$ satisfies $\cap_p F^p = 0$ (see Appendix D). For $\alpha \in H_T(N; \alpha)$ nonzero, define $p(\alpha)$ by $\alpha \in F^{p(\alpha)} - F^{p(\alpha)+1}$. It follows that

$$\alpha \cup \text{Eul}_T(\nu) = \alpha \cup \beta_0 \text{ modulo } F^{p+1}$$

which is nonzero. □

Notice that Definition 3.2 ensures that if N is a component of M^S and ν is a subbundle of the normal bundle of N in M , then the hypotheses of Lemma 3.9 apply.

Lemma 3.10. *Under the hypotheses of Theorem 3.3, for a generic choice of $\xi \in \mathfrak{t}$, the moment map component $\mu^\xi : M \rightarrow \mathbb{R}$ is η -equivariantly perfect.*

Proof. For generic choice $\xi \in \mathfrak{t}$, the image of $\exp : \text{span}\{\xi\} \rightarrow T$ is dense. Letting $f = \mu^\xi$, it follows that

$$\text{Crit}(f) = \{p \in M \mid \xi_M = 0\} = M^T$$

By Lemma 3.8, it suffices to show for each connected component C_i of M^T with negative normal bundle $\nu_i \rightarrow C_i$, that $H(C_i; \eta)$ possesses no $\text{Eul}_T(\nu_i)$ -torsion. This is a consequence of Lemma 3.9 in the case $S = T$. \square

Proof of Theorem 3.3. By Proposition 2.17, equivariant formality is equivalent to surjectivity of the natural map $H_T(M; \eta) \rightarrow H(M; \eta)$. We prove this by induction on $H(M_t; \eta)$ where $M_t := f^{-1}((-\infty, t))$ and f a generic component of the moment map as in Lemma 3.10.

For the base case M_t is empty for small t because M is compact.

In the induction step, assume that $H_T(M_{c_i-\epsilon}; \eta) \rightarrow H(M_{c_i-\epsilon}; \eta)$ is surjective. Using long exact sequences for the pair and Lemma 3.10 we obtain a commutative diagram:

$$(3.5) \quad \begin{array}{ccccccc} 0 & \longrightarrow & H_T(M_{c_i+\epsilon}, M_{c_i-\epsilon}; \eta) & \longrightarrow & H_T(M_{c_i+\epsilon}; \eta) & \longrightarrow & H_T(M_{c_i-\epsilon}; \eta) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & H(M_{c_i+\epsilon}, M_{c_i-\epsilon}; \eta) & \longrightarrow & H(M_{c_i+\epsilon}; \eta) & \longrightarrow & H(M_{c_i-\epsilon}; \eta) \end{array}$$

By a diagram chase we are reduced to proving that $H_T(M_{c_i+\epsilon}, M_{c_i-\epsilon}; \eta) \rightarrow H(M_{c_i+\epsilon}, M_{c_i-\epsilon}; \eta)$ surjects. By the Thom isomorphism this is equivalent to showing that the critical sets C_i are equivariantly formal. By the compatibility of η this follows from Example 2.5.

The injectivity of $H_T(M; \eta) \rightarrow H_T(M^T; \eta)$ follows from 2.15, because $H_T(M; \eta)$ is torsion free. \square

Corollary 3.11. *Under the hypotheses of 3.3 we have an equality*

$$\dim H(M; \eta(0)) = \dim H(M^T; \eta(0))$$

Proof. By equivariant formality

$$H_T(M; \eta) \cong H(M; \eta(0)) \otimes \hat{S}\mathfrak{t}^*$$

while by equivariant perfection of a generic component of the Morse map

$$\text{gr}(H_T(M; \eta)) \cong \oplus H(C_i; \eta(0)) \otimes \hat{S}\mathfrak{t}^*$$

if we ignore the \mathbf{Z}_2 -grading. The summands $H(C_i; \eta(0)) \otimes \hat{S}t^*$ are free, hence projective over $\hat{S}t^*$ so

$$H_T(M; \eta) \cong H(\cup_i C_i; \eta(0)) \otimes \hat{S}t^* = H(M^T; \eta(0)) \otimes \hat{S}t^*.$$

□

We now turn our attention to the proof of the Kirwan surjectivity Theorem 3.4. We will need the following proposition.

Proposition 3.12. *Let M be a compact T -manifold with nondegenerate abstract moment map $\mu : M \rightarrow \mathfrak{t}^*$ for which $0 \in \mathfrak{t}^*$ is a regular value, and suppose that M admits a T -invariant almost complex structure. Then we may choose a basis ξ_1, \dots, ξ_n of \mathfrak{t} such that*

- (1) *each $\mathfrak{t}_k := \text{Span}\{\xi_1, \dots, \xi_k\}$ exponentiates to a rank k torus T_k*
- (2) *$0 \in \mathfrak{t}_k^*$ is a regular value for the moment map $\mu_k = \text{proj}_{\mathfrak{t}_k^*} \circ \mu$*
- (3) *The restriction of $\mu^{\xi_{k+1}}$ to the submanifold $M_k = \mu_k^{-1}(0) \subset M$ is Morse-Bott with critical set equal to the points where T_{k+1} acts with positive dimensional stabilizer.*

The proof of Proposition 3.12 is postponed until Appendix B.

Proposition 3.12 allows us to factor the Kirwan map $H_T(M; \eta) \rightarrow H_T(\mu^{-1}(0))$ through the sequence of submanifolds determined by Proposition 3.12

$$H_T(M) \rightarrow H_T(\mu_1^{-1}(0)) \rightarrow H_T(\mu_2^{-1}(0)) \dots \rightarrow H_T(\mu_n^{-1}(0)) = H_T(\mu^{-1}(0)).$$

Our strategy to prove Theorem 3.4 is to show that each map in this composition is surjective. We do this by applying the following lemma to the T -manifold $\mu_k^{-1}(0)$ with function $(\mu^{\xi_{k+1}})^2$, which completes the proof of Theorem 3.4.

Lemma 3.13. *Let X be compact smooth T -manifold with no orbits of dimension smaller than d , and let N be the union of dimension d orbits. Let $f : X \rightarrow \mathbb{R}$ be an T -invariant function such that $\text{Crit}(f) = N \cup f^{-1}(c_0)$, where c_0 is the minimum value of f . Suppose that f is Morse-Bott except possibly at the minimum $f^{-1}(0)$. For $C = C_i$ a connected component of N , let \mathfrak{t}_C denote the infinitesimal stabilizer of C and let $T_C = \exp(\mathfrak{t}_C)$ its torus. If $\eta \in \Omega_T^3(X)$ is a d_T -closed form satisfying*

$$\eta|_C \in \Omega_{T/T_C}(C) \otimes 1 \subset \Omega_{T/T_C}(C) \otimes St_C^* \cong \Omega_T(C),$$

then the map induced by inclusion

$$H_T(X; \eta) \rightarrow H_T(f^{-1}(c_0); \eta)$$

is surjective.

Proof. The map $H_T(X_{c_0+\epsilon}; \eta) \rightarrow H_T(f^{-1}(c_0); \eta)$ is an isomorphism, hence surjective.

Now suppose inductively that $H_T(X_{c_i-\epsilon}; \eta) \cong H_T(X_{c_{i-1}+\epsilon}; \eta) \rightarrow H_T(f^{-1}(c_0); \eta)$ is surjective for some i . We must show that $H_T(X_{c_i+\epsilon}; \eta) \rightarrow H_T(X_{c_i-\epsilon}; \eta)$ is surjective. By Lemma 3.8, it will suffice to show that $\text{Eul}_T(v_i)$ is not a zero

divisor for $H_T(C_i; \eta)$, where C_i is a connected component of N . This follows from Lemma 3.9 using $S = T_{C_i}$. \square

Proof of Theorem 3.4. \square

4. GENERALIZED COMPLEX GEOMETRY

Let V be an n dimensional vector space. There is a natural bi-linear pairing of signature (n, n) on $V \oplus V^*$ which is defined by

$$\langle X + \alpha, Y + \beta \rangle = \frac{1}{2}(\beta(X) + \alpha(Y)).$$

A **generalized complex structure** on a vector space V is an orthogonal linear map $\mathcal{J} : V \oplus V^* \rightarrow V \oplus V^*$ such that $\mathcal{J}^2 = -1$. Let $L \subset V_{\mathbb{C}} \oplus V_{\mathbb{C}}^*$ be the $\sqrt{-1}$ eigenspace of the generalized complex structure \mathcal{J} . Then L is maximal isotropic and $L \cap \bar{L} = \{0\}$. Conversely, given a maximal isotropic $L \subset V_{\mathbb{C}} \oplus V_{\mathbb{C}}^*$ so that $L \cap \bar{L} = \{0\}$, there exists an unique generalized complex structure \mathcal{J} whose $\sqrt{-1}$ eigenspace is exactly L .

Let M be a manifold. A **generalized almost complex structure** on a manifold M is an orthogonal bundle map $\mathcal{J} : TM \oplus T^*M \rightarrow TM \oplus T^*M$ such that for any $x \in M$, \mathcal{J}_x is a generalized complex structure on the vector space $T_x M$.

Given a closed three form $H \in \Omega^3(M)$, an **H-twisted generalized complex structure** \mathcal{J} is a generalized almost complex structure such that the sections of the $\sqrt{-1}$ eigenbundle of \mathcal{J} are closed under the η -**twisted Courant bracket**, i.e., the bracket defined by the formula

$$[X + \xi, Y + \zeta] = [X, Y] + L_X \zeta - L_Y \xi - \frac{1}{2} d(\zeta(X) - \xi(Y)) + \iota_Y \iota_X H.$$

A generalized almost Kähler structure is a pair of two commuting generalized almost complex structures $\mathcal{J}_1, \mathcal{J}_2$ such that $\langle -\mathcal{J}_1 \mathcal{J}_2 \xi, \xi \rangle > 0$ for any $\xi \neq 0 \in C^\infty(TM \oplus T^*M)$, where $\langle \cdot, \cdot \rangle$ is the canonical pairing on $TM \oplus T^*M$. A generalized almost Kähler structure $(\mathcal{J}_1, \mathcal{J}_2)$ is called an H-twisted generalized Kähler structure if both \mathcal{J}_1 and \mathcal{J}_2 are H-twisted generalized complex structures. Given a generalized almost Kähler structure $(\mathcal{J}_1, \mathcal{J}_2)$, define $\mathcal{G}(A, B) := \langle -\mathcal{J}_1 \mathcal{J}_2 A, B \rangle$, $A, B \in C^\infty(TM \oplus T^*M)$. Then \mathcal{G} is a Riemannian metric on $TM \oplus T^*M$, and its restriction to TM defines a Riemannian metric g on M . Let $G = -\mathcal{J}_1 \mathcal{J}_2$. Since $G^2 = \text{id}$, $TM \oplus T^*M = C_+ \oplus C_-$, where C_\pm is the ± 1 -eigen-bundle of G . Let $\pi : TM \oplus T^*M \rightarrow TM$ be the projection map. Then

$$\pi|_{C_\pm} : C_\pm \rightarrow TM$$

is an isomorphism. Since \mathcal{J}_1 commutes with G , C_\pm is invariant under \mathcal{J}_1 . By projecting from C_\pm , \mathcal{J}_1 induces two almost complex structure I_+ and I_- on TM which are compatible with the Riemannian metric g .

Conversely, if there are two almost complex structures I_+ and I_- which are compatible with a Riemannian metric g on M , then

$$(4.1) \quad \mathcal{J}_{1/2} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} I_+ \pm I_- & -(\omega_+^{-1} \mp \omega_-^{-1}) \\ \omega_+ \mp \omega_- & -(I_+^* \pm I_-^*) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix}$$

is a generalized almost Kähler structure, where $\omega_{\pm} = gI_{\pm}$ are the fundamental 2-forms of the Hermitian structures (g, I_{\pm}) , and b is a two form. The following remarkable result is due to Gualtieri.

Theorem 4.1. ([Gua03]) *An H-twisted **generalized Kähler structure** is equivalent to a triple (g, I_+, I_-) consisting of a Riemannian metric g and two integrable almost complex structure compatible with g , satisfying the integrability conditions:*

$$d_+^c \omega_+ + d_-^c \omega_- = 0, \quad H + db = d_+^c \omega_+, \quad dd_{\pm}^c \omega_{\pm} = 0,$$

where $\omega_{\pm} = gI_{\pm}$, d_{\pm}^c are the $i(\bar{\partial} - \partial)$ operator associated to the complex structure I_{\pm} , and b is a two form. In particular, a triple (g, I_+, I_-) satisfying the above assumption defines a generalized Kähler pair $\mathcal{J}_1, \mathcal{J}_2$ by the formula (4.1).

We close this section with a quick review of generalized complex submanifolds as introduced in [BB03] (See also [BS06]). Although [BB03] only defined generalized complex submanifolds for untwisted generalized complex structures, the definition given there extends naturally to the twisted case as well.

Let W be a submanifold of an η -twisted generalized complex manifold (M, \mathcal{J}) , let $L \subset T_{\mathbb{C}}M \oplus T_{\mathbb{C}}^*M$ be the $\sqrt{-1}$ -eigenbundle of \mathcal{J} , and let $i : W \rightarrow M$ be the inclusion map. At each point $x \in N$ set

$$L_{W,x} = \{X + (\xi|_{T_{\mathbb{C}}W}) : X + \xi \in L \cap (T_{\mathbb{C},x}W \oplus T_{\mathbb{C},x}^*M)\}.$$

This defines a maximally isotropic distribution of $T_{\mathbb{C}}W \oplus T_{\mathbb{C}}^*W$ whose sections are closed under the i^* H-twisted Courant bracket. If L_W is a subbundle of $T_{\mathbb{C}}W \oplus T_{\mathbb{C}}^*W$ and if $L_W \cap \bar{L}_W = 0$, then W is said to be a generalized complex submanifold.¹ It is clear from the definition that if W is a generalized complex submanifold then there exists a unique i^* H-twisted generalized complex structure \mathcal{J}_W on W whose $\sqrt{-1}$ -eigenbundle is exactly L_W .

It is well-known that the fixed point submanifold of a symplectic torus action on a symplectic manifold is a symplectic submanifold. [Lin06] extends this fact to generalized complex manifolds.

Proposition 4.2. *Suppose the action of a torus T on an H-twisted generalized complex manifold (M, \mathcal{J}) preserves the generalized complex structure \mathcal{J} . And suppose Z is a connected component of the fixed point set. Then Z is a generalized complex submanifold of M . Let $i : Z \rightarrow M$ be the inclusion map. Then Z carries a i^* H-twisted generalized complex structure.*

¹ It is noteworthy that the sufficient and necessary conditions for W to be a generalized complex submanifolds have been found in [BS06].

5. GENERALIZED MOMENT MAPS

First we recall the definition of Hamiltonian actions on H -twisted generalized complex manifolds given in [LT05].

Definition 5.1. ² [LT05]) *Let a compact Lie group G with Lie algebra \mathfrak{g} act on a manifold M , preserving an H -twisted generalized complex structure \mathcal{J} , where $H \in \Omega^3(M)^G$ is closed. The action of G is said to be Hamiltonian if there exists a smooth equivariant function $\mu : M \rightarrow \mathfrak{g}^*$, called the **generalized moment map**, and a 1-form $\alpha \in \Omega^1(M, \mathfrak{g}^*)$, called the **moment one form**, so that*

- a) $\mathcal{J}d\mu^\xi = -\xi_M - \alpha^\xi$ for all $\xi \in \mathfrak{g}$, where ξ_M denotes the induced vector field.
- b) $H + \alpha$ is an equivariantly closed three form in the usual Cartan Model.

Remark 5.2. An H -twisted generalized complex structure $\mathcal{J} : TM \oplus T^*M \rightarrow TM \oplus T^*M$ induces by restriction and projection a map $\beta : T^*M \rightarrow TM$ which is a real Poisson bi-vector, see for instance [Gua03] and [BS06]. If the action of a compact Lie group G on a generalized complex manifold (M, \mathcal{J}) is Hamiltonian with a generalized moment map $\mu \rightarrow \mathfrak{g}^*$, then a straightforward calculation shows

$$-\beta(d\mu^\xi) = \xi_M,$$

where ξ_M is the vector field on M induced by $\xi \in \mathfrak{g}$. This shows clearly that the action of G is Hamiltonian with respect to the Poisson bi-vector β .

Let a compact Lie group G act on a twisted generalized complex manifold (M, \mathcal{J}) with generalized moment map μ . Let \mathcal{O}_a be the co-adjoint orbit through $a \in \mathfrak{g}^*$. If G acts freely on $\mu^{-1}(\mathcal{O}_a)$, then \mathcal{O}_a consists of regular values and $M_a = \mu^{-1}(\mathcal{O}_a)/G$ is a manifold, which is called the **generalized complex quotient**. The following two results were proved in [LT05].

Lemma 5.3. *Let a compact Lie group G act freely on a manifold M . Let H be an invariant closed three form and let α be an equivariant mapping from \mathfrak{g} to $\Omega^1(M)$. Fix a connection $\theta \in \Omega(M, \mathfrak{g}^*)$. Then if $H + \alpha \in \Omega_G^3(M)$ is equivariantly closed, there exists a natural form $\Gamma \in \Omega^2(M)^G$ so that $\iota_{\xi_M} \Gamma = \alpha^\xi$. Thus $H + \alpha + d_G \Gamma \in \Omega^3(M)^G \subset \Omega_G^3(M)$ is closed and basic and so descends to a closed form $\tilde{H} \in \Omega^3(M/G)$ so that $[\tilde{H}]$ is the image of $[H + \alpha]$ under the Kirwan map.*

Proposition 5.4. *Assume there is a Hamiltonian action of a compact Lie group G on an H -twisted generalized complex manifold (M, \mathcal{J}) with generalized moment map $\mu : M \rightarrow \mathfrak{g}^*$ and moment one-form $\alpha \in \Omega^1(M, \mathfrak{g}^*)$. Let \mathcal{O}_a be a co-adjoint orbit through $a \in \mathfrak{g}^*$ so that G acts freely on $\mu^{-1}(\mathcal{O}_a)$. Given a connection on $\mu^{-1}(\mathcal{O}_a)$, the generalized complex quotient M_a inherits an \tilde{H} -twisted generalized*

²Indeed, Condition (b) was not imposed in [LT05, Definition A.2.]. However, in order to make the quotient construction work, Tolman and the author made it clear in [LT05, Prop. A.7, A.10] that $H + \alpha$ must be equivariantly closed in the usual equivariant Cartan model.

complex structure $\tilde{\mathcal{J}}$, where \tilde{H} is defined as in the Lemma 5.3. Up to B-transform, $\tilde{\mathcal{J}}$ is independent of the choice of connection.

5.1. Nitta's theorem and compatibility. The rest of this section is devoted to the proof of Proposition 1.1, which indeed has already been established by Nitta in [NY07]. However, since Proposition 1.1 is central to our paper and since we believe more details are needed in Nitta's argument to make it more accessible, we will present a self-contained detailed proof in our paper. The essential step in the proof of Proposition 1.1 is the non-trivial observation that the restriction of α^ξ to the fixed point set F^ξ vanishes. To prove it in full generality, we are going to use the maximum principle of pseudo-holomorphic functions on almost complex manifolds, as advocated in [NY07, Prop. 3.1]. Note that our proof differs slightly from the one given in [NY07]. For instance, our proof does not involve the use of a Levi-Civita connection, and we apply Proposition 4.2 in an essential way. We would also like to mention that when the generalized complex manifolds have constant types, one can construct more elementary proofs using the Darboux theorem of generalized complex structures [Gua03].

Lemma 5.5. *Suppose the trivial action of a torus T on a compact H -twisted generalized complex manifold (M, \mathcal{J}) is Hamiltonian with a generalized moment map μ and a moment one form α . Then $d\mu^\xi = \alpha^\xi = 0$ for all $\xi \in \mathfrak{t}$.*

Proof. It has been shown that there exists a generalized almost complex structure \mathcal{J}_2 such that $\mathcal{J}_1 = \mathcal{J}$ and \mathcal{J}_2 form a generalized almost Kähler pair, see for instance, [Ca06, Sec. 3] and [NY07]. As we explained in Section 4, the generalized almost Kähler structure induces a triple (g, I_+, I_-) consisting of a Riemannian metric g and two almost complex structures I_+ and I_- compatible with g ; moreover, one can reconstruct \mathcal{J}_1 and \mathcal{J}_2 from the triple (g, I_+, I_-) using Formula (4.1). Given $\xi \in \mathfrak{t}$, by assumption we have $\mathcal{J}_1 d\mu^\xi = \alpha^\xi$, i.e.,

$$(5.1) \quad \frac{1}{2} \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} I_+ + I_- & -(\omega_+^{-1} - \omega_-^{-1}) \\ \omega_+ - \omega_- & -(I_+^* + I_-^*) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix} \begin{pmatrix} 0 \\ d\mu^\xi \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha^\xi \end{pmatrix},$$

where $\omega_\pm = gI_\pm$. A straightforward calculation shows that

$$\omega_+^{-1}(d\mu^\xi) = \omega_-^{-1}(d\mu^\xi), \quad I_+^* d\mu + I_-^* d\mu^\xi = 2\alpha^\xi.$$

Since I_\pm are compatible with g , we have $\omega_\pm = gI_\pm = -I_\pm^* g$ and so $\omega_\pm^{-1} = g^{-1} I_\pm^*$. It follows from $\omega_+^{-1}(d\mu^\xi) = \omega_-^{-1}(d\mu^\xi)$ that $I_+^* d\mu^\xi = I_-^* d\mu^\xi$. Thus $I_\pm^* d\mu^\xi = \alpha^\xi$. However, since by assumption the trivial action is Hamiltonian, condition b) in Definition 5.1 implies that $d\alpha^\xi = 0$. Locally, we can always find a function h such that $I_\pm^* d\mu^\xi = dh$. If the generalized almost complex structure \mathcal{J}_2 is integrable, i.e., $(\mathcal{J}_1, \mathcal{J}_2)$ forms a generalized Kähler pair, then I_\pm must be integrable complex structures and μ^ξ is locally the real part of the I_\pm -holomorphic function $\mu^\xi + \sqrt{-1}h$. If M is

compact, it follows from the maximum principle of the real part of a holomorphic function that μ^ξ has to be a constant. In the general case, locally μ^ξ is the real part of a pseudo-holomorphic function with respect to almost complex structures I_\pm . By the maximum principle of the real part of a pseudo-holomorphic function as we explained in Appendix A, μ^ξ has to be a constant provided M is compact. \square

Now consider the Hamiltonian action of a compact connected torus T on a compact H -twisted generalized complex manifold M with a generalized moment map μ . Suppose that $\xi \in \mathfrak{t}$ generates a compact connected subtorus T_1 in T , i.e., ξ is a generic element in the Lie algebra \mathfrak{t}_1 of $T_1 \subset T$.

Lemma 5.6. *Under the above assumptions, the critical set*

$$\text{Crit}(\mu^\xi) = \{x \in M \mid (d\mu^\xi)_x = 0\}$$

coincides with the fixed point set F of the T_1 action on M for any $\xi \in \mathfrak{g}$.

Proof. The inclusion $\text{Crit}(\mu^\xi) \subset F$ is obvious. It suffices to show that for any $x \in F$ we have $(d\mu^\xi)_x = 0$. By Proposition 4.2 the fixed point set F of the T_1 -action is a generalized complex submanifold. Moreover, it follows from [Lin07, Lemma 4.8] that the induced trivial action of T_1 on F is Hamiltonian with the generalized moment map $\mu|_F: F \rightarrow \mathfrak{t}_1^*$. Since M is compact, F has to be compact itself. By Lemma 5.5, we have $(d\mu^\xi)|_F = 0$. Choose a T_1 -invariant metric on M . Since μ^ξ is T_1 -invariant, $\text{grad}(\mu^\xi)$, the gradient flow of μ^ξ , is also invariant under the linearized T_1 action. It follows that at each point $x \in F$, $\text{grad}(\mu^\xi)$ is tangent to F . Thus $\langle \text{grad} \mu^\xi, d\mu^\xi \rangle_x = 0$. This implies that for any $x \in F$, $(d\mu^\xi)_x = 0$. \square

Remark 5.7. It is clear from the proof that in the statement of Lemma 5.6, we need only to assume that all fixed points submanifolds are compact. This compactness assumption here is essential. For instance, Hu [Hu05, Sec. 4.5] constructed an example of a Hamiltonian S^1 action on a generalized complex manifold with a proper moment map f such that $\text{crit}(f) \not\subseteq M^{S^1}$. Lemma 5.6 fails because in Hu's example the fixed point submanifold is a copy of complex plane \mathbb{C} which is non-compact.

We are ready to give a proof of Proposition 1.1. We are going to use the same notations as in Lemma 5.6.

Proof of Proposition 1.1. Let $f := \mu^\xi$. In view of Lemma 5.6, it suffices to show that the Hessian of f is nondegenerate in the normal direction to M^{T_1} .

Because ξ is generic, $M^{T_1} = \ker(X)$. The vector field X linearizes at $p \in M^{T_1}$ to $\mathcal{A} \in \text{End}(T_p M)$ defined by the formula $\mathcal{A}(w_p) = [X, w]_p$ for $w \in C^\infty(TM)$. Since T_1 is connected we have $T_p(M^{T_1}) = \ker \mathcal{A}$ as subsets of $T_p M$. The Hessian of f at p is a linear map $\text{Hess}_p(f) : T_p M \rightarrow T_p^* M$, defined

by $\text{Hess}_p(f)(w_p) = (dL_w(f))_p$ where $w \in C^\infty(TM)$. We need to show that for all $p \in \text{crit}(f)$, $\ker(\text{Hess}_p(f)) \subset T_p \text{crit}(f)$.

Now let β be the canonical Poisson bivector associated to the generalized complex structure \mathcal{J} . As we explained in Remark 5.2, we have $X = -\beta df$. Thus

$$-[X, w] = L_w X = -L_w(\beta df) = -(L_w \beta) df - \beta(L_w df).$$

Thus,

$$\mathcal{A}(w_p) = (L_w \beta)_p df_p + \beta_p(L_w df)_p = \beta_p(L_w df)_p = \beta_p(\text{Hess}_p(f)(w_p)),$$

where we've used that $df_p = 0$ and $(L_w df)_p = (dL_w f)_p = \text{Hess}_p(f)(w_p)$. Thus $\ker(\text{Hess}_p(f)) \subset \ker(\mathcal{A}) = T_p(M^{T_1}) = T_p \text{crit}(f)$ as desired. \square

Remark 5.8. Choose an invariant generalized almost complex structure \mathcal{J}_2 such that $(\mathcal{J}, \mathcal{J}_2)$ form a generalized almost Kähler pair. It is easy to show that for any $p \in M$, $T_p M$ splits as the direct sum of $T_p M^{T_1}$ and N , where N is the orthogonal complement of $T_p M^{T_1}$ in $T_p M$ with respect to the Riemannian metric induced by the generalized almost Kähler pair $(\mathcal{J}, \mathcal{J}_2)$; moreover, the vector space N inherits a generalized Kähler structure which is invariant under the linearized action of T_1 on N . It follows that N admits a complex structure which is invariant under the operator \mathcal{A} which we defined in the proof of Proposition 1.1. As a direct consequence, we see that the $\text{Hess}_p(f)$ must have even index. So $f = \mu^\xi$ must be a Morse-Bott function of even index.

It follows easily from Proposition 1.1 that the twisting form $H + \alpha$ is compatible with the torus action (Definition 3.2).

Corollary 5.9. *Let $T \times M \rightarrow M$ be a Hamiltonian T -action for a compact, connected H -twisted generalized complex manifold M with moment map $\mu : M \rightarrow \mathfrak{t}^*$ and moment 1-form α . For $x \in M$, denote \mathfrak{t}_x to be the infinitesimal stabilizer of x . Then*

$$\ker(\alpha_x) \supseteq \mathfrak{t}_x$$

where we regard α_x as an element of $\text{Hom}(\mathfrak{t}, T_x^* M)$.

Proof. Condition a) of Definition 5.1 asserts

$$-\xi_M - \alpha^\xi = \mathcal{J} d\mu^\xi$$

for all $\xi \in \mathfrak{t}$. Proposition 1.1 says that if $(\xi_M)_x = 0$ then $d\mu_x^\xi = 0$ and consequently $\alpha_x^\xi = 0$. \square

6. KIRWAN INJECTIVITY AND SURJECTIVITY AND ITS APPLICATION

Proposition 1.1 establishes that the moment map μ of a compact H -twisted generalized Hamiltonian T -space is a nondegenerate abstract moment map and Corollary 5.9 establishes that the equivariant twisting 3-form $H + \alpha$ is compatible. Thus Theorems 1.3 and 1.4 follow from Theorem 3.3 and Theorem 3.4 respectively. Theorem 1.2 then follows from Proposition 2.18

because the restriction of a generalized Hamiltonian action is generalized Hamiltonian.

One of the early motivations for this paper was to prove the following result.

Theorem 6.1. *Let (M, J, H) be a complex generalized complex manifold with nonexact twisting H . Then any generalized Hamiltonian torus action must have fixed point locus of dimension at least four.*

Proof. Let T be the torus and μ the moment map. By Proposition 1.1 and Corollary 5.9 (M, T, μ) is an abstract moment map with compatible equivariant twisting $H + \alpha$ where α is the moment 1-form. Thus by Corollary 3.11,

$$\dim H(M; H) = \dim H(M^T; H).$$

On the other hand, 0 is always a compatible twisting so

$$\dim H(M) = \dim H(M^T).$$

Since H is not exact, we know by (2.4) that $\dim H(M; H) < \dim H(M)$. Consequently $H(M^T; H) < H(M^T)$ and we conclude that the restriction of H to M^T is *not* exact. Since H is a 3-form, it must be that M^T has a component of dimension 3 or more, and because M^T is even dimensional (see Prop. 4.2) this completes the proof. \square

Theorem 6.1 stands in stark contrast with the symplectic world, where Hamiltonian actions with isolated fixed points abound (e.g. toric manifolds). In the course of writing this paper, we discovered a proof of Theorem 6.1 that avoids twisted cohomology and in fact leads to even stronger constraints. We will present these arguments in a future paper, along with new examples of compact GC Hamiltonian actions.

APPENDIX A. MAXIMUM PRINCIPLE FOR PSEUDO-HOLOMORPHIC FUNCTIONS ON ALMOST COMPLEX MANIFOLDS

In this section, we give a self-contained proof of the maximum principle for pseudo-holomorphic functions on almost complex manifolds. We believe that the maximum principle in this setting should have been known to experts working in the related areas and we are not claiming any originality. We are presenting a proof here just because the central results of our paper are built upon it for which we can not find a good reference.

Let (M, J) be an almost complex manifold. Then the almost complex structure induces a splitting of the complexified cotangent bundle $T_{\mathbb{C}}^*(M) = T^*(M)^{1,0} \oplus T^*(M)^{0,1}$. In this context, a complex valued function $f + ig \in C^\infty(M)$ is defined to be a pseudo-holomorphic function on M if $(df)_x + i(dg)_x \in T_x^*(M)^{1,0}$ for any $x \in M$. In this appendix, we prove the following result.

Theorem A.1. *Suppose (M, J) is an almost complex manifold. Suppose f is the real part of a pseudo-holomorphic function $f + ig$ on M . Then for any $x \in M$ there exists an open neighborhood $B \ni x$ such that*

$$\sup_B f = \sup_{\partial B} f, \quad \inf_B f = \inf_{\partial B} f.$$

Before beginning the proof, we first recall the maximum principle for the elliptic partial differential equations of second order as treated in [GT1977]. Let L be a second order linear differential operator on a domain Ω of \mathbb{R}^m given by

$$(A.1) \quad Lu = a^{ij} D_{ij} u + b^i D_i u + c_i u,$$

where $a^{ij} = a^{ji}$, $D_i u = \frac{\partial u}{\partial x_i}$, $D_{ij} u = \frac{\partial^2 u}{\partial x_i \partial x_j}$. L is said to be elliptic at $x \in \Omega$ if the coefficient matrix $[a^{ij}(x)]$ is positive definite. L is said to be elliptic in Ω if it is elliptic at each point of Ω . The following maximum principle is a fundamental result in the theory of elliptic operators.

Theorem A.2. ([GT1977, Thm. 3.1]) *Let L be an elliptic operator in the bounded domain Ω . Suppose that*

$$(A.2) \quad Lu \geq 0 (\leq 0) \text{ in } \Omega, \quad c = 0 \text{ in } \Omega,$$

with $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$. Then

$$\sup_{\Omega} u = \sup_{\partial \Omega} u, \quad \inf_{\Omega} u = \inf_{\partial \Omega} u.$$

We are ready to present a proof of Theorem A.1.

Proof. Given an arbitrary point $p \in M$, we can choose a coordinate neighborhood $(U, x^1, x^2, \dots, x^{2n})$ around p such that under this coordinate system the almost complex $J(x) = [J_j^i(x)]$ coincides with the standard complex structure on \mathbb{R}^{2n} at the point p , i.e.,

$$J(p) = \begin{bmatrix} 0_n & I_n \\ -I_n & 0_n \end{bmatrix},$$

where 0_n denotes the $n \times n$ zero matrix and I_n the $n \times n$ identity matrix. Note that $\frac{\partial}{\partial x_k} + iJ \frac{\partial}{\partial x_k} \in C^\infty(T(M)^{0,1})$ for any $1 \leq k \leq 2n$. We have

$$\left\langle \frac{\partial}{\partial x_k} + iJ \frac{\partial}{\partial x_k}, df + idg \right\rangle = 0$$

since $df + idg \in C^\infty(T^*(M)^{1,0})$.

Observe $J \frac{\partial}{\partial x_k} = J_k^p \frac{\partial}{\partial x_p}$. We get the following generalized Riemann-Cauchy equations.

$$(A.3) \quad \frac{\partial f}{\partial x_k} = J_k^p \frac{\partial g}{\partial x_p}, \quad \frac{\partial g}{\partial x_k} = -J_k^p \frac{\partial f}{\partial x_p}$$

Therefore

$$\begin{aligned}
\frac{\partial}{\partial x_k} \left(\frac{\partial f}{\partial x_k} \right) &= \frac{\partial}{\partial x_k} \left(J_k^p \frac{\partial g}{\partial x_p} \right) \\
&= \frac{\partial J_k^p}{\partial x_k} \frac{\partial g}{\partial x_p} + J_k^p \frac{\partial}{\partial x_p} \left(\frac{\partial g}{\partial x_k} \right) \\
&= \frac{\partial J_k^p}{\partial x_k} J_p^q \frac{\partial f}{\partial x_q} + J_k^p \frac{\partial}{\partial x_p} \left(-J_k^q \frac{\partial f}{\partial x_q} \right) \\
&= \frac{\partial J_k^p}{\partial x_k} J_p^q \frac{\partial f}{\partial x_q} - J_k^p \frac{\partial J_k^q}{\partial x_p} \frac{\partial f}{\partial x_q} - J_k^p J_k^q \frac{\partial^2 f}{\partial x_p \partial x_q}.
\end{aligned}$$

It follows

$$\frac{\partial^2 f}{\partial x_k^2} + J_k^p J_k^q \frac{\partial^2 f}{\partial x_p \partial x_q} + \left(J_k^p \frac{\partial J_k^q}{\partial x_p} - \frac{\partial J_k^p}{\partial x_k} J_p^q \right) \frac{\partial f}{\partial x_q} = 0.$$

Summing over the index k we get a second order linear equation

$$(A.4) \quad \sum_k \left(\frac{\partial^2 f}{\partial x_k^2} + J_k^p J_k^q \frac{\partial^2 f}{\partial x_p \partial x_q} \right) + \sum_k \left(J_k^p \frac{\partial J_k^q}{\partial x_p} - \frac{\partial J_k^p}{\partial x_k} J_p^q \right) \frac{\partial f}{\partial x_q} = 0.$$

Set

$$a^{pq} = \delta_p^q + \sum_k J_k^p J_k^q,$$

where δ_p^q is the Kronecker symbol. Since $J(p)$ coincides with the standard complex structure on \mathbb{R}^{2n} , a simple calculation shows that the matrix $[a^{pq}(p)] = 2I_{2n}$, where I_{2n} denotes the $2n \times 2n$ identity matrix. This shows clearly that the matrix $[a^{pq}]$ is a positive definite symmetric matrix at p and so must be a positive definite symmetric matrix on an open ball $p \in B \subset U$. Thus Equation A.4 is a second order elliptic equation on a bounded ball B . Now Theorem A.1 is a simple consequence of Theorem A.2. \square

APPENDIX B. NONDEGENERATE ABSTRACT MOMENT MAPS

In this section we will prove Lemma 3.12. This result was used without much explanation in appendix G of [GGK02] but was later recognized to be subtler than it appears. Here we provide a detailed proof using an extra hypothesis, the existence of an invariant almost (more generally stable) complex structure.

Throughout, let T be a compact torus with lie algebra \mathfrak{t} , M be a smooth T -manifold, and $\phi : M \rightarrow \mathfrak{t}^*$ a nondegenerate, abstract moment map (Def. 3.1).

As explained in [GGK02], condition 1 of Definition 3.1 is equivalent to the condition that for all $p \in M$,

$$(B.1) \quad d\phi_p(T_p M) = \mathfrak{t}_p^\perp.$$

Here given a subspace $\mathfrak{h} \subset \mathfrak{t}$, $\mathfrak{h}^\perp \subset \mathfrak{t}^*$ denotes the annihilator of $\mathfrak{h} \subset \mathfrak{t}$. That said, the next Lemma is not so surprising:

Lemma B.1. *Let $p, q \in M$ satisfy $\mathfrak{t}_p = \mathfrak{t}_q = \mathfrak{g}$ where \mathfrak{g} is the Lie algebra of a subtorus $G \subset T$. If p and q lie in the same connected component of M^G then*

$$\phi(p) + \mathfrak{t}_p^\perp = \phi(q) + \mathfrak{t}_q^\perp$$

Proof. The map $\phi_{\mathfrak{g}} : M \rightarrow \mathfrak{g}^*$ defined by composing ϕ with the projection $\pi_{\mathfrak{g}^*} : \mathfrak{t}^* \rightarrow \mathfrak{g}^*$ is a moment map for the restricted G action. By definition, $\phi_{\mathfrak{g}}$ restricts to a locally constant function on M^G so $\phi_{\mathfrak{g}}(p) = \phi_{\mathfrak{g}}(q)$ and so $\phi(p) - \phi(q) \in \ker(\pi_{\mathfrak{g}^*}) = \mathfrak{g}^\perp$. \square

For compact T manifold M , there can only be a finite number of distinct isotopy groups $T_p \subset T$, for each of which M^{T_p} has a finite number of components. We deduce:

Corollary B.2. *If M is compact, then the set of vector spaces*

$$\{\text{span}(\phi(p)) + \mathfrak{t}_p^\perp \mid p \in M\}$$

is finite.

Lemma B.3. *Let M be a compact T -manifold equipped with a nondegenerate moment map ϕ and suppose that $0 \in \mathfrak{t}^*$ is a regular value for ϕ . There exists a codimension 1 subtorus $H \subset T$ with Lie algebra \mathfrak{h} for which 0 is a regular value for $\phi_{\mathfrak{h}} = \pi_{\mathfrak{h}^*} \circ \phi$, where $\pi_{\mathfrak{h}^*} : \mathfrak{t}^* \rightarrow \mathfrak{h}^*$ is projection.*

Proof. The hyperplane Grassmanian $\text{Gr}_1(\mathfrak{t})$ parametrizes the set of codimension one subspaces $\mathfrak{h} \subset \mathfrak{t}$. Those subspaces integrating to codimension one subtori form a dense subset of $\text{Gr}_1(\mathfrak{t})$, so to prove Lemma B.3 it will suffice to show that the set

$$\mathcal{U} := \{\mathfrak{h} \in \text{Gr}_1(\mathfrak{t}) \mid 0 \text{ is a regular value for } \phi_{\mathfrak{h}} = \text{proj}_{\mathfrak{h}^*} \circ \phi\}$$

contains a nonempty open set.

It is somewhat clearer to work with the projective space $P(\mathfrak{t}^*)$, which is canonically isomorphic to $\text{Gr}_1(\mathfrak{t})$ via the correspondence $\mathfrak{h} \leftrightarrow \mathfrak{h}^\perp$. Since $\mathfrak{h}^\perp = \ker(\pi_{\mathfrak{h}^*})$ it follows easily that 0 is a regular value for $\phi_{\mathfrak{h}}$ if and only if

$$\mathfrak{t}^* = \text{im}(d\phi_p) + \mathfrak{h}^\perp = \mathfrak{t}_p^\perp + \mathfrak{h}^\perp$$

for all $p \in M$ satisfying $\phi(p) \in \mathfrak{h}^\perp$.

If $\phi(p) = 0$ then $\text{im}(d\phi_p) = \mathfrak{t}^*$ by hypothesis. If $\phi(p) \in \mathfrak{h}^\perp - 0$, then $\text{span}(\phi(p)) = \mathfrak{h}^\perp$. Thus if \mathfrak{h}^\perp lies outside of the finite set of *proper* vector subspaces described in Corollary B.2, the moment map $\phi_{\mathfrak{h}}$ is guaranteed to be regular at zero. This is an open and nonempty condition, completing the proof. \square

Iterating Lemma B.3 enables us to prove (i) and (ii) of Proposition 3.12. In particular, we construct a sequence of subtori $T = T_n \supseteq T_{n-1} \supseteq T_{n-1} \dots \supseteq$

T_1 such that the moment map ϕ_k for each T_k is regular at 0, and then choose arbitrarily $\xi_k \in \mathfrak{t}_k - \mathfrak{t}_{k-1}$.

We prove (iii) in two steps. We denote the restricted function $f_k := \phi^{\xi_{k+1}}|_{\phi_k^{-1}(0)}$.

Lemma B.4. *The critical set $\text{Crit}(f_k) = \{p \in \phi_k^{-1}(0) \mid (t_{k+1})_p \neq 0\}$.*

Proof. For $p \in M$ we have

$$\dim(d_p f_k(T_p \phi_k^{-1}(0))) = \dim(d_p \phi_{k+1}(T_p \phi_k^{-1}(0))) = \dim(d_p \phi_{k+1}(T_p M)) - k = 1 - \dim((t_k)_p).$$

ere we have used Equality B.1. \square

It remains to prove that the critical points of $f_k := \phi^{\xi_{k+1}}|_{\phi_k^{-1}(0)}$ are nondegenerate. According to [GGK02], if M admits an invariant almost complex structure, then in the neighborhood of every orbit M admits a symplectic structure for which the moment map is Hamiltonian. We can then use the following local canonical form [GS82] for symplectic Hamiltonian actions.

Lemma B.5. ([GS82]) *Suppose that M admits a T -invariant almost complex structure. For $p \in M$ choose a complimentary Lie subalgebra $\mathfrak{h} \subset \mathfrak{t}$ to \mathfrak{t}_p so that $\mathfrak{t}^* = \mathfrak{h}^* \oplus \mathfrak{t}_p^*$. Then for some T_p representation V , there is a T -equivariant diffeomorphism from an invariant neighborhood of p to an invariant neighborhood of the zero section of the associated bundle $T \times_{T_p} (\mathfrak{h}^* \oplus V)$ sending the moment map ϕ to the map*

$$\phi' : T \times_{T_p} (\mathfrak{h}^* \oplus V) \rightarrow \mathfrak{h}^* \oplus \mathfrak{t}_p^* = \mathfrak{t}^*$$

defined by $\phi'(t, \eta, v) = (\eta, q(v))$, where $q : V \rightarrow \mathfrak{t}_p^$ is a quadratic form.*

Applying this to the case $T = T_{k+1}$, $\phi = \phi_{k+1}$, $\mathfrak{h} = \mathfrak{t}_k$ at a critical point p of f_k , we obtain a local model for ϕ_{k+1} near p in M

$$\phi' : T_{k+1} \times_{(T_{k+1})_p} (\mathfrak{t}_k^* \oplus V) \rightarrow \mathfrak{t}_k^* \oplus \mathbb{R} = \mathfrak{t}_{k+1}^*$$

where $\phi'(t, \eta, v) = \eta + q(v)$. Here a neighborhood of p in $\mu_k^{-1}(0)$ maps to $T_{k+1} \times_{(T_{k+1})_p} (\{0\} \oplus V)$ and f_k corresponds (up to a nonzero scalar multiple) to the quadratic form q . Thus in some local coordinates, f_k looks like a quadratic form near p in $\phi_k^{-1}(0)$ and hence is nondegenerate, completing the proof of Lemma 3.12.

APPENDIX C. FURTHER ASPECTS OF TWISTED EQUIVARIANT COHOMOLOGY

We call a G -invariant open cover $\{U_\alpha \mid \alpha \in I\}$ of M equivariantly good if all nonempty intersections of the U_α are tubular neighborhoods of some G -orbit in M . For example, every G -equivariant vector bundle over a compact manifold M admits a *finite* equivariantly good cover.

Lemma C.1. *Let M be a G -manifold admitting a finite equivariantly good open cover $\{U_\alpha | \alpha \in I\}$. Then for all $J \subset I$, and for any twisting d_G -closed 3-form $\eta \in \Omega_G^3(M)$ we have an isomorphism of $H_G(M)$ -modules*

$$H_G(\cap_{\alpha \in J} U_\alpha; \eta) \cong H(BH)$$

for some closed subgroup $H \subset G$.

Proof. By definition we know that $\cap_{\alpha \in J} U_\alpha$ is equivariantly homotopy equivalent to a homogeneous space G/H for some subgroup $H \subset G$. Thus η , $H_G(\cap_{\alpha \in J} U_\alpha) \cong H(G/H)$. But it is a standard result that $H_G(G/H) \cong H(BH)$ and $H^3(BH) = 0$ (c.f. [AB84]). Thus η is cohomologous to zero. It follows that

$$H_G(\cap_{\alpha \in J} U_\alpha; \eta) \cong H_G(\cap_{\alpha \in J} U_\alpha) \cong H(BH)$$

□

Lemma C.1 can be used to prove twisted equivariant cohomology results using Mayer-Vietoris.

Proposition C.2. *Let M be a smooth G -manifold admitting a finite, equivariantly good open cover. Then for any twisting η , $H_G(M; \eta)$ is a finitely generated $(\hat{S}\mathfrak{g}^*)^G$ -module.*

Proof. For any closed subgroup $H \subset G$, $H(BH)$ is finitely generated over $H(BG) \cong (\hat{S}\mathfrak{g}^*)^G$ (the number of generators is bounded by the Weyl group). Then $H_G(M; \eta)$ is shown to be finitely generated by repeated application of Mayer-Vietoris and Lemma D.2. □

There are a couple of different versions of twisted equivariant cohomology described in the literature and we take a moment to reconcile them. In [FHT02], it is defined as the cohomology of the complex of formal Laurent series $\Omega_G^*(M)((\beta))$ where β has degree -2 with differential $d + \eta\beta$. This makes $\Omega_G^*(M)((\beta))$ into a graded complex, producing \mathbb{Z} -graded cohomology $\tilde{H}_G^*(M; \eta)$. The reader can readily verify that $\sum_k \alpha^{n+2k} \otimes \beta^k \in \Omega_G^n(M)((\beta))$ is closed (exact) if and only if $\sum_k \alpha^{n+k} \in \hat{\Omega}_G(M)$ is closed (exact) for $d_G + \eta$. It follows that

$$\tilde{H}_G^n(M) \cong H_G^{[n]}(M)$$

where on the n is an integer and $[n]$ is its reduction mod 2.

Another version comes from [HuU06]. They define $H_G(M; \eta)$ to be the cohomology of the complex $(\Omega(M) \otimes \hat{S}\mathfrak{g}^*)$ with differential $d_G + \eta\wedge$. There is a natural injective chain map

$$(\Omega(M) \otimes \hat{S}\mathfrak{g}^*) \hookrightarrow \hat{\Omega}_G(M)$$

which induces an isomorphism in cohomology in most interesting cases:

Proposition C.3. *Let M be a smooth G -manifold admitting a finite, equivariantly good open cover. Then $\tilde{H}_G(M) \cong H_G(M)$.*

Proof. Using Equation 2.1, it is easy to prove the isomorphism holds on tubular neighborhoods of G -orbits. This extends to M by repeated application of Mayer-Vietoris. \square

This result allows us to compare $H_G(M; \eta)$ with the cohomology of the direct sum complex $(\Omega_G(M) = \bigoplus_i \Omega_G^i(M), d_G + \eta \wedge)$.

Proposition C.4. *For M admitting a finite equivariantly good cover, $H_G(M; \eta)$ is canonically isomorphic to $H(\Omega_G(M), d_G + \eta) \otimes_{(S\mathfrak{g}^*)^G} (\hat{S}\mathfrak{g}^*)^G$ (i.e. it is obtained by extension of scalars).*

Proof. By Proposition C.3 $H_G(M; \eta) \cong \bar{H}_G(M; \eta)$. The formal power series ring $\hat{S}\mathfrak{g}^*$ is flat over $S\mathfrak{g}$ so it factors through taking cohomology (see Lemma D.5). Thus

$$\bar{H}_G(M; \eta) = H(\Omega_G(M) \otimes_{(S\mathfrak{g}^*)^G} (\hat{S}\mathfrak{g}^*)^G; d + \eta) = H(\Omega_G(M); d + \eta) \otimes_{(S\mathfrak{g}^*)^G} (\hat{S}\mathfrak{g}^*)^G.$$

\square

Recall ([Mc01] Def. 3.8) that given a filtration $\{F^p A\}$ of a differential complex (A, d) , the associated spectral sequence (E_r^p, d_r) is said to **converge strongly** to $H(A, d)$ if the induced maps $F^p H(A, d) / F^{p+1} H(A, d) \rightarrow E_\infty^p$ and $H(A, d) \rightarrow (\varprojlim H(A, d) / F^p H(A, d))$ are isomorphisms. The first of these

Proposition C.5. *Let M admit a finite G -equivariantly good open cover. Then the spectral sequences associated to the filtrations $\{F^p\}$ and $\{L^p\}$ of $(\hat{\Omega}_G(M), d_{G, \eta})$ described in §2.3 both converge strongly to $H_G(M; \eta)$.*

Proof. First note that the filtrations are cofinal, satisfying

$$F^{2p-n} \subset L^p \subset F^{2p+n},$$

so convergence of the spectral sequence determined by $\{F^p\}$ implies convergence of that for $\{L^p\}$.

Clearly $\cup_p F^p = \hat{\Omega}_G(M)$ and $\cap_p F^p = 0$ so the filtration is exhaustive and weakly convergent. By ([Mc01] Thm 3.2) it only remains to show that we have an isomorphism

$$(C.1) \quad H_G(M; \eta) \cong \varprojlim H_G(M; \eta) / F^p H_G(M; \eta).$$

First note that if $\eta = 0$, then the filtration $F^p H_G(M) = \prod_{k \geq p} H_G^k(M)$ so (C.1) certainly holds. Next, note that wedging by $\exp(b)$ for any $b \in \Omega_G^2(M)$ preserves the filtration F^p , so by Lemma 2.12, (C.1) must hold for exact η . Finally, because η must become exact when restricted to tubular neighborhoods of orbits, we may use Mayer-Vietoris and the five lemma to prove the general case. \square

Proposition C.5 was proven with more general hypotheses in [FHT02] using a Mittag-Leffler condition. We include our own proof because it is more elementary.

APPENDIX D. COMMUTATIVE ALGEBRA

The results in this section are standard. Our principal reference is Matsumura [Mat89] Chapter 8.

Recall that a module is Noetherian if it satisfies the ascending chain condition. A ring is Noetherian if it is Noetherian over itself. Throughout, R will denote a Noetherian, commutative integral domain.

Lemma D.1. *Let M be an R -module. The following are equivalent:*

- (1) M is Noetherian
- (2) M is finitely generated
- (3) M has a finite presentation

Lemma D.2. *For a short exact sequence of R -modules*

$$0 \rightarrow K \rightarrow M \rightarrow N \rightarrow 0$$

M is Noetherian if and only if both K and N are Noetherian.

An R -module M is called flat if the functor $M \otimes_R : \text{Mod}_R \mapsto \text{Ab}$ from R -modules to abelian groups is exact.

Lemma D.3. *For R an Noetherian, integral domain, the following are flat over R :*

- (1) the quotient field of R .
- (2) any I -adic completion of R , for $I \subset R$ an ideal.
- (3) any projective module over R .

Recall that for $I \subset R$ an ideal and M an R -module, the I -adic completion of M is defined by the inverse limit:

$$\hat{M} = \varprojlim M/I^k M$$

We have \hat{R} is a ring and \hat{M} is a \hat{R} -module.

Lemma D.4. *If M is finitely generated R -module, then there is a natural isomorphism of \hat{R} -modules:*

$$\hat{M} \cong M \otimes_R \hat{R}$$

Recall that a R -chain complex (C, d) is an R -module C , equipped with a morphism $d : C \rightarrow C$ satisfying $d^2 = 0$. The homology is defined $H(C, d) = \ker d / \text{im } d$ and is an R -module. Given a module M over R , the tensor product $(C \otimes M, d \otimes 1)$ is a \mathbb{Z} -chain complex.

Lemma D.5. (ex. 7.6 in [Mat89]) *Let (C, d) be an R -chain complex and let M be flat over R . Then we have a natural isomorphism $H(C, d) \otimes_R M \cong H(C \otimes_R M, d \otimes_R 1)$.*

The Jacobson radical is the ideal

$$J := \{r \in R \mid 1 - rs \text{ is a unit in } R \text{ for all } s \in S\}.$$

Lemma D.6. (Nakayama's Lemma) *Let M be a nonzero finitely generated module over R and J the Jacobson radical of R . Then $JM \neq M$.*

We remark that Nakayama's Lemma works for not necessarily commutative rings (see 4.3.10 [We94]) so we can apply it to super commutative rings like $H_T(M)$.

We have some particular examples in mind. For instance the polynomial ring $A = \mathbb{R}[x_1, \dots, x_n]$. If $I = (x_1, \dots, x_n) \subset A$ is the augmentation ideal, then the I -adic completion is $\hat{A} = \mathbb{R}[[x_1, \dots, x_n]]$, the ring of formal power series, so \hat{A} is flat over A . Both A and \hat{A} are Noetherian commutative integral domains so their quotient fields are flat over each of them.

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